4) The Parabola
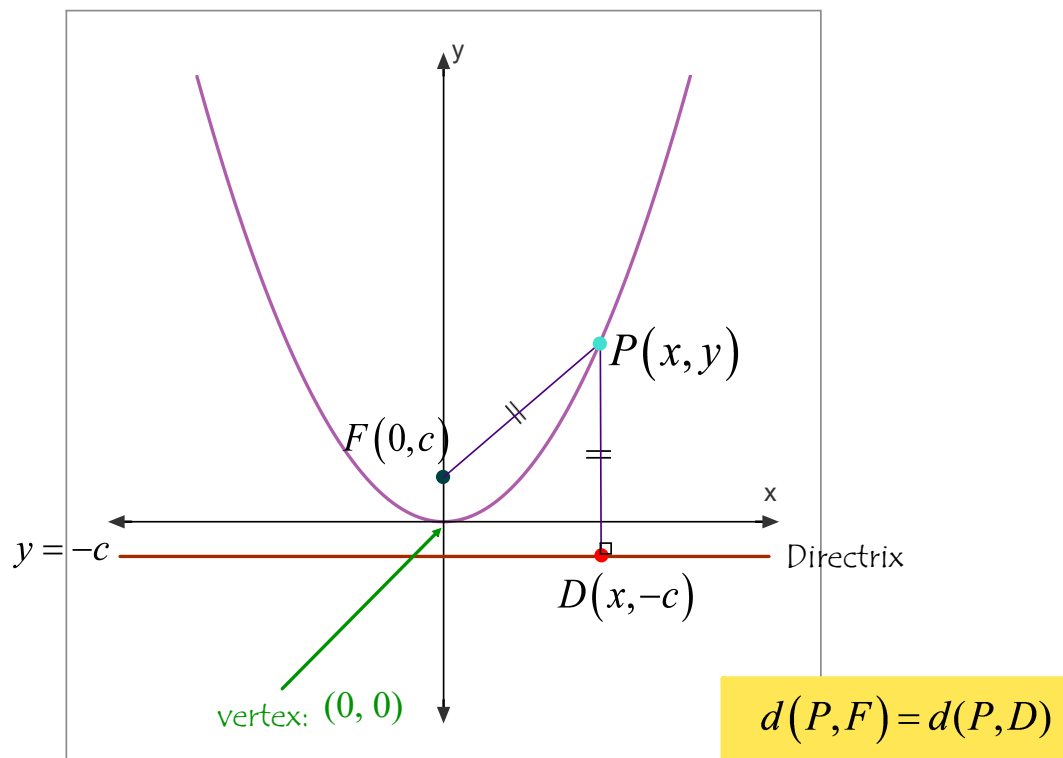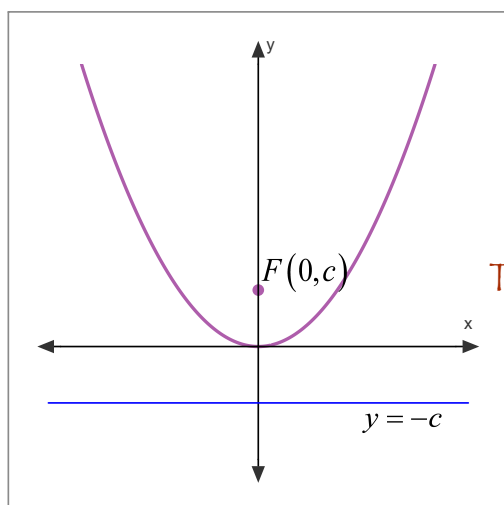
The locus of all points that are equidistant from a fixed point (called a focus) and a fixed line (called a directrix).

$F(0,c)$

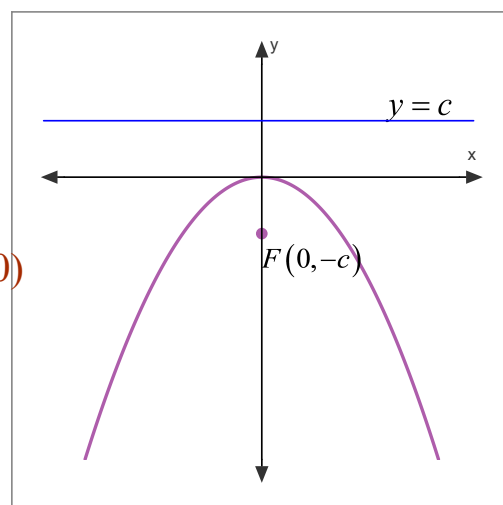$P(x,y)$

$y = -c$

$D(x,-c)$

Directrix

x

y

vertex: $(0, 0)$

$$d(P,F) = d(P,D)$$

Vertical
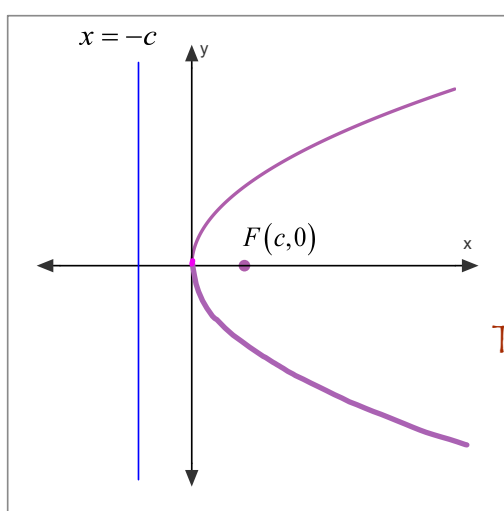

Equation

$$x^2 = \pm 4cy$$

The vertex is $(0,0)$

$x^2 = 4cy$

$x^2 = -4cy$

c is the distance from the vertex to the focus or the distance from the vertex to the directrix.
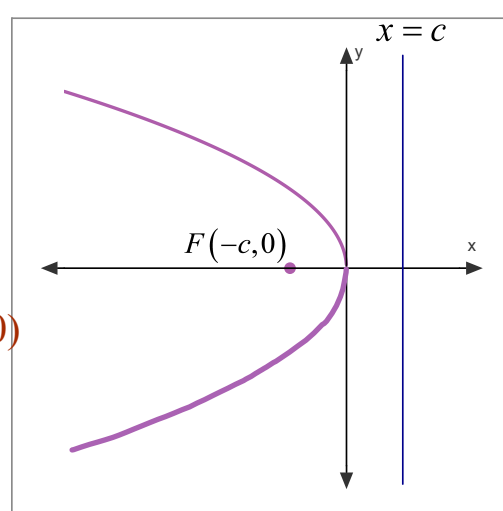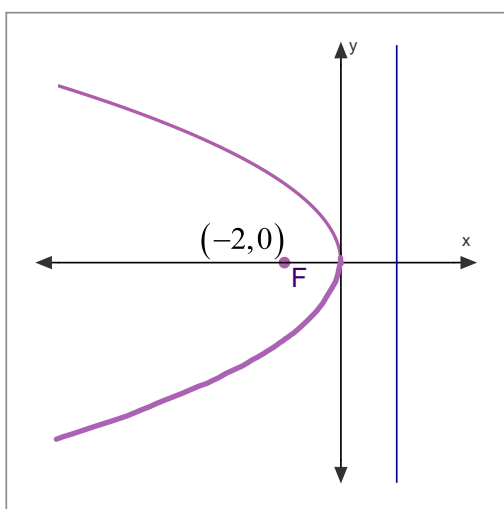
Horizontal

$x = -c$

$F(c,0)$

$$y^2 = 4cx$$

Equation

$$y^2 = \pm 4cx$$

The vertex is $(0,0)$

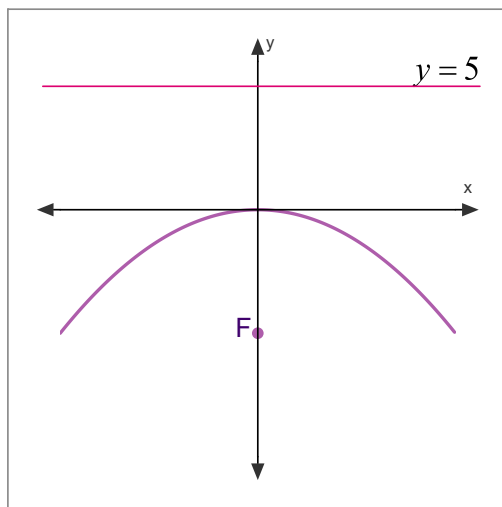$x = c$

$F(-c,0)$

$$y^2 = -4cx$$

Example:

Determine:

a) the equation of the parabola

b) the equation of the directrix

Example:

Determine:
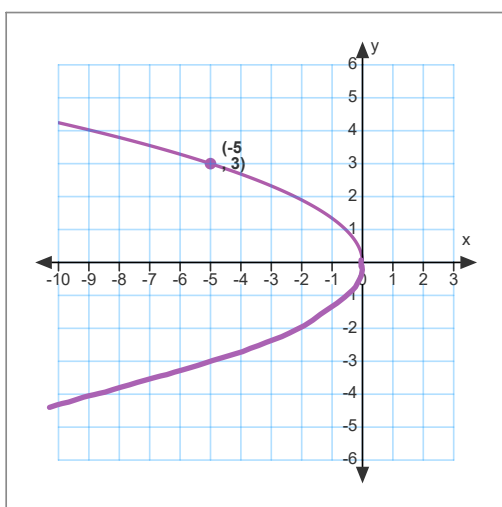a) the equation of the parabola
b) the coordinates of the focus

Example:

Given $y^2 = 24x$

a) determine the vertex.

b) determine the focus.

c) determine the equation of the directrix.

d) if $P(x,5)$ lies on the parabola, determine x.

Example:　　For $x^2 = -32y$
　　　　　　　a) determine y in the point $P(2, y)$
　　　　　　　b) determine the coordinates of the focus
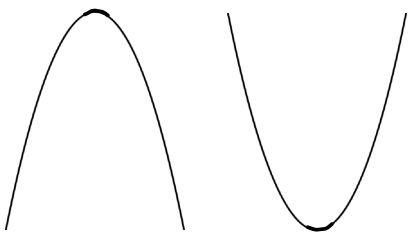
Example:

Find:

a) the equation of the parabola
b) the coordinates of the focus
c) the equation of the directrix

Transformed Parabola

- vertex is now (h, k)

The equations become (standard form):

Vertical

Horizontal / Sideways

$$(x-h)^2 = \pm 4c(y-k)$$

$$(y-k)^2 = \pm 4c(x-h)$$

Example:

A graph showing a parabola opening upward with vertex $V(1,-1)$, focus $F(1,2)$, and directrix $y=-4$.
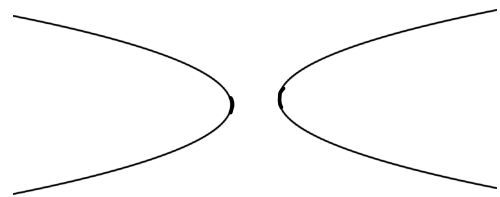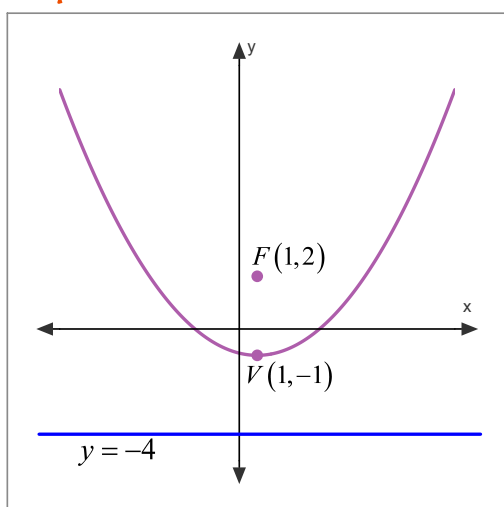
Determine the equation of the parabola.
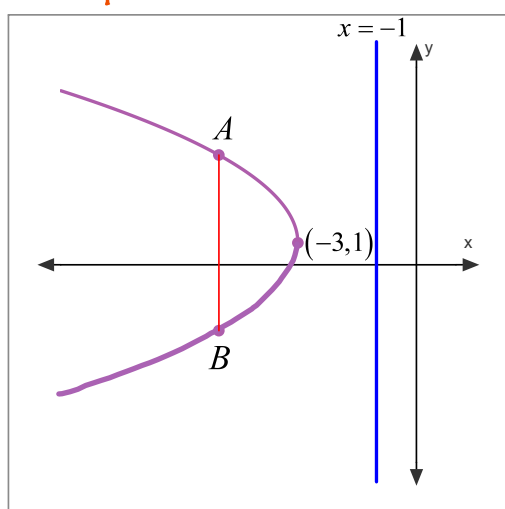
Example:

Determine:
a) the equation of the parabola
b) the length of segment AB if it passes through the focus.

Example:

Determine the equation of the parabola whose vertex is (2, –1) and whose focus is (4, –1)

Example: Write the equation of
a) $y^2 = 6x$ b) $x^2 = -4y$
after they have undergone a translation of
$t(3, -1)$.

Standard Form & General Form

- Standard to general

$$(x+5)^2 = 3(y-2)$$ ⟶ Standard form

Expand

$$x^2 + 10x + 25 = 3y - 6$$

$$x^2 + 10x - 3y + 31 = 0$$ ⟶ General form

Write the following equation of a parabola in general form.

$$(y-1)^2 = 6\left(x - \frac{1}{2}\right)$$

- General to standard

$$x^2 - 4x - 2y - 4 = 0$$ General form

$$x^2 - 4x = 2y + 4$$

Complete the square on the LHS

$$x^2 - 4x + 4 = 2y + 4 + 4$$

Factor

$$(x - 2)^2 = 2y + 8$$

Factor the coefficient of the varibale on the RHS

$$(x - 2)^2 = 2(y + 4)$$ Standard form
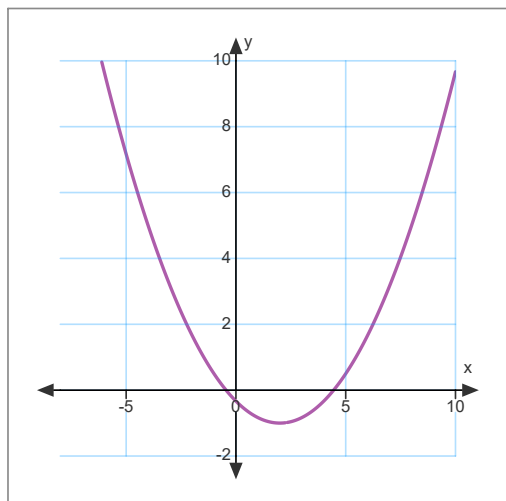
Change from general to standard

$$y^2 + 8x - 10y + 9 = 0$$

Example: Determine the coordinates of the focus and the equation of the directrix for the parabola whose equation is $2x^2 - 12x - 8y + 10 = 0$.
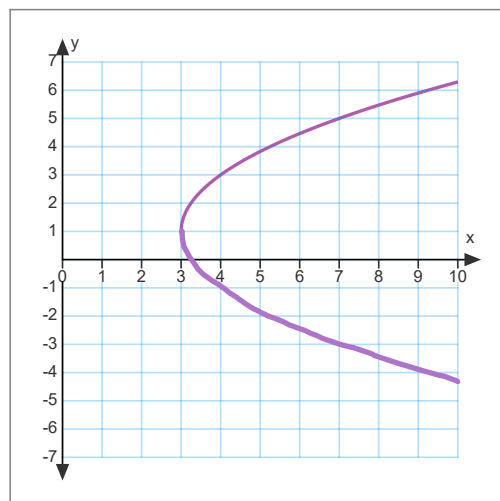
Example: Determine the equation of the parabola whose focus is at (2, –3) and whose directrix is at $y = 1$.
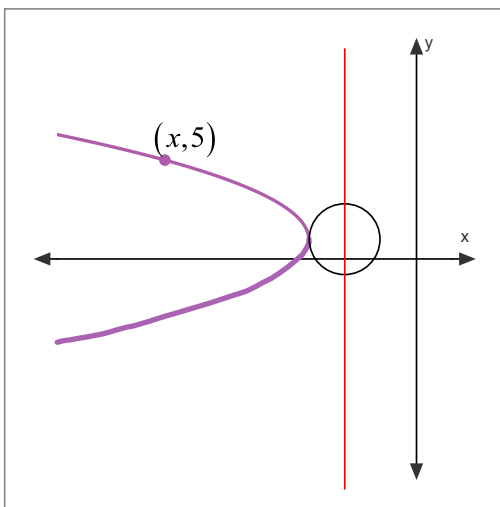
Parabolas and Inequalities

$$(x-2)^2 \geq 6(y+1)$$

$$(y-1)^2 \geq 4(x-3)$$

Example: In the diagram below,

the equation of the circle is

$$x^2 + y^2 + 4x - 2y + 4 = 0$$

The parabola is tangent to the circle at its vertex.

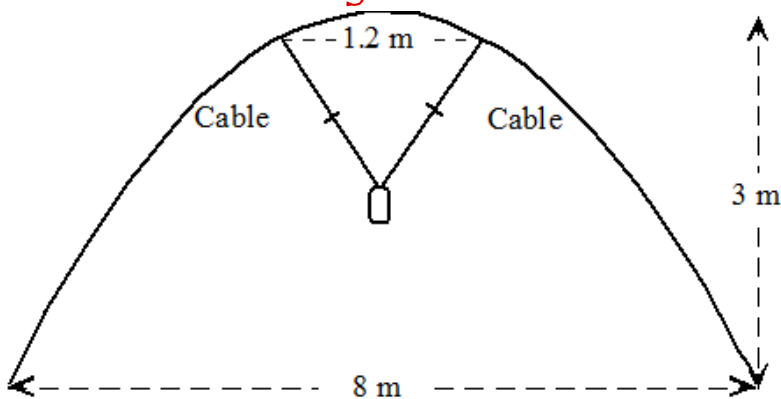The vertex of the parabola and the centre of the circle are on the same line. The directrix of the parabola passes through the centre of the circle.

What is the value of x in the point $(x, 5)$?

Example: A bridge is supported by an arch in the shape of a parabola. The distance between points A and B at the waterline is 60^m. The maximum height of the arch is 25^m. The top of a sailboat mast is 8^m above the surface of the water. What is the minimum distance from point A (or B) that the sailboat can pass under the arch?
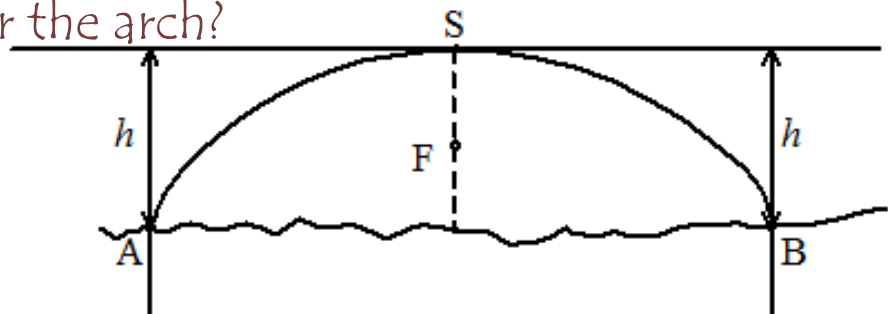
Example: A light is suspended at the focus of a parabolic arch by two cables of equal length. The arch is 8^m across its base and 3^m high. The cables are attached 1.2^m apart. Find the length of each cable.

Example: In a circus act, a lion has to leap through a circular ring. A chain fastens the ring to a metal support that is parabolic in shape. The following information is known:

- Point A is located at the vertex of the parabola and point B coincides with the focus.
- Chain AB has the same measure as the radius of the ring.
- The equation of the circle representing the ring is
$$(x-2)^2 + (y+6)^2 = 9$$

What is the equation of the parabola?
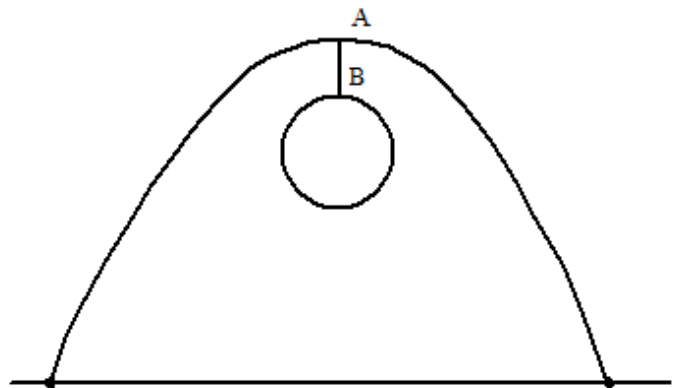
Example: A cylinder 200cm tall has a hole punched in its side, halfway up its height. When the cylinder is full of water, the stream that flows out follows a parabolic path whose equation is $x^2 = -12y$. After 5 minutes, the stream follows a path whose equation is $x^2 = -y$. Find the distance between the points where the two streams of water hit the ground.