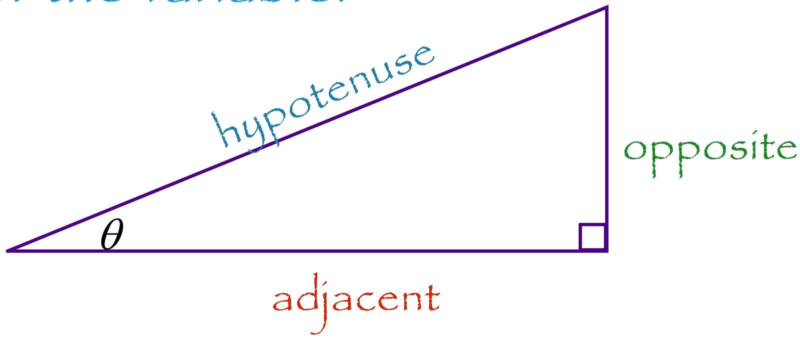
Trigonometric Identities

An identity is a statement that is true for all possible values of the variable.

Recall:



$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

Secant: $$\sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent}$$

Cosecant: $$\csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{opposite}$$
(cosec θ)

Cotangent: $$\cot\theta = \frac{1}{\tan\theta} = \frac{adjacent}{opposite}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

* $\cos^2\theta + \sin^2\theta = 1$

$i)$ $\cos^2\theta = 1 - \sin^2\theta$

$ii)$ $\sin^2\theta = 1 - \cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\Downarrow$$

* $1+\tan^2\theta=\sec^2\theta$

$i)\ \tan^2\theta=\sec^2\theta-1$

$ii)\ \sec^2\theta-\tan^2\theta=1$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$* \quad \cot^2\theta + 1 = \csc^2\theta$$

i) $\cot^2\theta = \csc^2\theta - 1$

ii) $\csc^2\theta - \cot^2\theta = 1$

We use identities to simplify statements or in proofs by using substitution .

Examples: Simplify each of the following expressions.

a) $\sin x \sec x \cot x = \frac{\cancel{opp}}{\cancel{hyp}} \cdot \frac{\cancel{hyp}}{\cancel{adj}} \cdot \frac{\cancel{adj}}{\cancel{opp}} = 1$

$\cancel{\sin x} \cdot \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} = 1$

b) $\frac{\tan^2\theta}{1+\tan^2\theta} = \frac{\tan^2\theta}{\sec^2\theta}$

① $= \frac{\sec^2\theta - 1}{\sec^2\theta} = 1 - \cos^2\theta = \sin^2\theta$

② $\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} = \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \frac{\cancel{\cos^2\theta}}{1} = \sin^2\theta$

c) $\dfrac{\cos\theta + \sin^2\theta\sec\theta}{\sec\theta}$

$\dfrac{\cos\theta}{\sec\theta} + \sin^2\theta$

$\dfrac{\cos\theta}{\frac{1}{\cos\theta}} + \sin^2\theta$

$\cos\theta \cdot \dfrac{\cos\theta}{1}$

$\rightarrow \cos\theta \cdot \cos\theta + \sin^2\theta$

$\cos^2\theta + \sin^2\theta$

1

d) $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{\cos\theta}{1-\sin\theta}$

$\longrightarrow 1-\sin^2\theta=\cos^2\theta$

$\cancel{1-\sin\theta=\cos\theta}$

$\left(\dfrac{1-\sin\theta}{1-\sin\theta}\right)\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{\cos\theta}{1-\sin\theta}\left(\dfrac{1+\sin\theta}{1+\sin\theta}\right)$

$\dfrac{\cos\theta-\cos\theta\cancel{\sin\theta}+\cos\theta+\cos\theta\cancel{\sin\theta}}{1-\sin^2\theta}$

$=\dfrac{2\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}=\dfrac{2\cancel{\cos\theta}}{\cancel{\cos\theta}\cdot\cos\theta}$

$=\dfrac{2}{\cos\theta}$ $\rightarrow 2\cdot\dfrac{1}{\cos\theta}=\boxed{2\sec\theta}$

Examples: Prove each of the following identities.

a) $\left(\cos\theta - \sin^2\theta\cos\theta\right)\left(1+\tan^2\theta\right) = \cos\theta$

Simplify the left side.

$\left(\cos\theta\left(1 - \sin^2\theta\right)\right)\left(\sec^2\theta\right) = \cos\theta$

$\cos\theta\left(\cancel{\cos^2\theta}\right)\cdot\frac{1}{\cancel{\cos^2\theta}} = \cos\theta$

$\cos\theta = \cos\theta$

b) $$\frac{\sec\theta\csc\theta}{\tan\theta+\cot\theta}=1$$

$$\frac{\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}=1$$

$$\frac{\frac{1}{\cos\theta\cdot\sin\theta}}{\left(\frac{\sin\theta}{\sin\theta}\right)\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}=1$$

$$\frac{\frac{1}{\cos\theta\sin\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\cdot\sin\theta}}=1$$

$$\frac{\frac{1}{\cos\theta\sin\theta}}{\frac{1}{\cos\theta\cdot\sin\theta}}=1$$

$$1=1$$